4) Primjenom Kramerovog pravila riješiti sistem

$$\begin{cases} 2x+y+z=6 \\ x-y+2z=15 \\ -x-y+3z=16 \end{cases}$$

Rj. $D=\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & -1 & 3 \end{vmatrix} = -6-2-1-1-3+4 = -9 \neq 0$

$D_1=\begin{vmatrix} 6 & 1 & 1 \\ 15 & -1 & 2 \\ 16 & -1 & 3 \end{vmatrix} = -18+32-15+16+45+12 = -18$

$D_2=\begin{vmatrix} 2 & 6 & 1 \\ 1 & 15 & 2 \\ -1 & 16 & 3 \end{vmatrix} = 2\begin{vmatrix} 15 & 2 \\ 16 & 3 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 15 \\ -1 & 16 \end{vmatrix} =$

$= 2(45-32)-6(3+2)+16+15 = 27$

$D_3 = -45$

4

$\Rightarrow \exists!$ rj. $x=\frac{D_1}{D}=\frac{-18}{-9}=2$, $y=\frac{D_2}{D}=\frac{27}{-9}=-3$, $z=\frac{D_3}{D}=\frac{-45}{-9}=5$

$\Rightarrow$ Rj. sistema je $(2,-3,5)$.

3) Riješiti sistem j-na:

$$\begin{cases} x-2y+z=1 \\ 2x+3y-z=2 \\ 4x-y+z=4 \end{cases}$$

Rj. $D=\begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{vmatrix} = 0$

$D_1=\begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{vmatrix} = 0 = D_2 = D_3$

$D=0 \wedge D_1=D_2=D_3=0 \Rightarrow$ dodatno diskutujemo.

<u>I način</u>: $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 2 & 3 & -1 & 2 \\ 4 & -1 & 1 & 4 \end{array}\right) \begin{matrix} (-2) \\ (-4) \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 7 & -3 & 0 \\ 0 & 7 & -3 & 0 \end{array}\right) (-1) \sim$

$\sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 7 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow r(A)=2 \wedge r(\bar{A})=2 \rightarrow$ sistem saf. neodređen:

$x-2y+z=1$

$7y-3z=0 \Rightarrow \boxed{y=\frac{3z}{7}}$

$x = 1+z+2\cdot\frac{3z}{7} = \boxed{\frac{7+13z}{7}}$

Rj. je $\left(\frac{7+13z}{7}, \frac{3z}{7}, z, z\in\mathbb{R}\right)$

<u>II način</u> $M(D)=\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0 \Rightarrow$ sistem je ekviv. sa prve dvije j-ne

$$\begin{cases} x-2y+z=1 \\ 2x+3y-z=2 \end{cases} \oplus \Rightarrow \quad 3x+y=3$$

$\boxed{y=3-3x}$

Opšte rj. je $(x, 3-3x, 7-7x)$, $x\in\mathbb{R}$

$x-2(3-3x)+z=1$

$x-6+6x+z=1 \Rightarrow \boxed{z=7-7x}$

6) Riješiti i diskutovati sistem:

$$\begin{cases} x+y+z=6 \\ mx+4y+z=5 \\ 6x+(m+2)y+2z=13 \end{cases}$$

R:

$$D=\begin{vmatrix} 1 & 1 & 1 \\ m & 4 & 1 \\ 6 & m+2 & 2 \end{vmatrix} = 8+6+m^2+\cancel{2m}-24-m-2-\cancel{2m} = m^2-m-12 =$$
$$=(m-4)(m+3).$$

$$D_1=\begin{vmatrix} 6 & 1 & 1 \\ 5 & 4 & 1 \\ 13 & m+2 & 2 \end{vmatrix} = 48+13+5(m+2)-(52+10+6m+12) = -(m+3)$$

$$D_2=\begin{vmatrix} 1 & 6 & 1 \\ m & 5 & 1 \\ 6 & 13 & 2 \end{vmatrix} = 10+36+13m-30-12m-13 = m+3$$

$$D_3=\begin{vmatrix} 1 & 1 & 6 \\ m & 4 & 5 \\ 6 & m+2 & 13 \end{vmatrix} = 6(m+3)(m-4).$$

1° $D\neq 0 \Rightarrow m\neq 4 \wedge m\neq -3.$

Sistem saglasan i određen.

$$x=\frac{D_1}{D}=\frac{-(\cancel{m+3})}{(m-4)(\cancel{m+3})}=\frac{-1}{m-4}\ ;\quad y=\frac{D_2}{D}=\frac{\cancel{m+3}}{(m-4)(\cancel{m+3})}=\frac{1}{m-4}.$$

$$z=\frac{D_3}{D}=\frac{6\cancel{(m+3)}\cancel{(m-4)}}{\cancel{(m-4)}\cancel{(m+3)}}=6 \Rightarrow \text{Jed. rj. je } \left(\frac{-1}{m-4}, \frac{1}{m-4}, 6\right)$$

2° $D=0 \Rightarrow m=4 \vee m=-3.$

2a) $m=4 \Rightarrow D_1\neq 0 \wedge D_2\neq 0 \Rightarrow$ nema rj.

2b) $m=-3 \Rightarrow D=0 \wedge D_1=D_2=D_3=0$

Potrebno je dodatno diskutovati.

~~Primjer~~ $\begin{cases} x+y+z=6 \\ -3x+4y+z=5 \\ 6x-y+2z=13 \end{cases}$ Jedan od minora je $\begin{vmatrix} 1 & 1 \\ -3 & 4 \end{vmatrix} = 7 \neq 0 \Rightarrow$ (5)

Sistem je ekvival. sa $\begin{cases} x+y+z=6 \\ -3x+4y+z=5 \end{cases} \Rightarrow$

$x = 6-y-z$

$-3(6-y-z)+4y+z=5$

$-18+3y+3z+4y+z=5 \Rightarrow y = \frac{23-4z}{7}$ Op. rj. je $\left(\frac{19-3z}{7}, \frac{23-4z}{7}, z\right)$

$z \in \mathbb{R}$

$x = 6 - \frac{23-4z}{7} - z = \frac{19-3z}{7}$.

(2)

2) Primjenom matrica riješiti sistem:

$$\begin{cases} x-3y+2z=-3 \\ 2x+y-3z=-13 \\ 3x+2y+z=4 \end{cases}$$

Rj. Neka je $A = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ i $b = \begin{pmatrix} -3 \\ -13 \\ 4 \end{pmatrix}$.

$A^{-1} / Ax = b \Rightarrow \boxed{X = A^{-1}b}$

$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

3) Primjenom Kramerovog pravila diskutovati i riješiti sistem u zavis. od parametra a:

$$\begin{cases} ax+y+z=1 \\ x+ay+z=2 \\ x+ay+az=-3 \end{cases}$$

Rj. $D = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & a & a \end{vmatrix} = \cancel{a^3+1+a-a-a-a^2} = \cancel{a^3-a^2-a+1}$

$a^3+1+1-a-a-a = a^3-3a+2 = a^3-a-2a+2 = a(a^2-1)-2(a-1) =$

$(a-1)(a^2+a-2) = (a-1)(a^2-a+2a-2) =$

$= a^2(a-1)-(a-1) = (a-1)(a-1)(a+1) =$

$= (a-1)^2(a+2)$

$D_x = \begin{vmatrix} 1 & 1 & 1 \\ 2 & a & 1 \\ -3 & a & a \end{vmatrix} = a^2-3+\cancel{2a}+3a-\cancel{2a}-a = a^2+2a-3 =$

$= a^2+a-2 = (a-1)(a+2)$

$D_y = \begin{vmatrix} a & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -3 & a \end{vmatrix} = 2a^2+1-3-2-a+3a = 2a^2+2a-4 =$

$= 2(a^2+a-2) = 2(a-1)(a+2)$

$$D_z = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 2 \\ 1 & 1 & -3 \end{vmatrix} = -3a^2+2+1-a-2a+3 = -3a^2-3a+6 =$$

$$= -3(a^2+a-2) = -3(a-1)(a+2)$$

1°) $D \neq 0 \Rightarrow (1-a)^2(a+2) \neq 0 \Leftrightarrow a \neq 1 \wedge a \neq -2$

$\rightarrow$ $D \neq 0 \rightarrow$ sist. ima jedin. rj.

3) $x = \frac{D_x}{D} = \frac{(a-1)(a+2)}{(1-a)^2(a+2)} = \frac{1}{a-1}$

$y = \frac{D_y}{D} = \frac{2(a-1)(a+2)}{(a-1)^2(a+2)} = \frac{2(a+2)}{(a-1)(a+2)} = \frac{2}{a-1}$

$z = \frac{D_z}{D} = \frac{-3(a-1)(a+2)}{(a-1)^2(a+2)} = \frac{-3}{a-1}$

Jed. rj. je $\left(\frac{1}{a-1}, \frac{2}{a-1}, \frac{-3}{a-1}\right)$.

2°) $D=0 \Rightarrow a=1 \vee a=-2$

2¹) $a=1 \Rightarrow D_x=D_y=D_z=0 \Rightarrow$ Potrebno je dodatno diskutovati

$\rightarrow$ Sistem glasi: $\left.\begin{array}{l} x+y+z=1 \\ x+y+z=2 \\ x+y+z=-3 \end{array}\right\}$ nemoguć! $\rightarrow$ nema rj.

2²) $D=0 \rightarrow a=-2 \rightarrow D_x=D_y=D_z=0$

$$\begin{cases} -2x+y+z=1 \\ x-2y+z=2 \\ x+y-2z=-3 \end{cases} \quad \left(\begin{array}{ccc|c} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \\ 1 & 1 & -2 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ -2 & 1 & 1 & 1 \\ 1 & 1 & -2 & -3 \end{array}\right) \begin{array}{l} 2 \\ (-1) \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -3 & 3 & 5 \\ 0 & 3 & -3 & -5 \end{array}\right) (-1) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -3 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

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~~Potrebno~~ $\begin{cases} x+y+z=6 \\ -3x+4y+z=5 \\ 6x-y+2z=13 \end{cases}$ Jedan od minora je $\begin{vmatrix} 1 & 1 \\ -3 & 4 \end{vmatrix} = 7 \neq 0 \Rightarrow$

Sistem je ekvival. sa $\begin{cases} x+y+z=6 \\ -3x+4y+z=5 \end{cases} \rightarrow$

$x = 6-y-z$

$-3(6-y-z)+4y+z=5$

$-18+3y+3z+4y+z=5 \Rightarrow y = \frac{23-4z}{7}$ Op. rj. je $\left(\frac{19-3z}{7}, \frac{23-4z}{7}, z\right)$

$z \in \mathbb{R}$

$x = 6 - \frac{23-4z}{7} - z = \frac{19-3z}{7}$.